SYMPLECTIC TORIC MANIFOLDS

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1. Introduction

A symplectic toric manifold is a symplectic manifold with a torus action satisfying certain nice conditions. We determine the homology groups of these manifolds by producing a Morse function from the moment map of the torus action and then using a result from Morse theory to get results in homology.

2. Important Objects

We begin by talking about what a symplectic toric manifold is, along with a few objects associated with such a manifold.

Definition 1. A *symplectic toric manifold* $(M, \omega, \mathbb{T}^n, \mu)$ is a compact connected symplectic manifold (M, ω) along with a faithful hamiltonian action of $\mathbb{T}^n$ on M, where $\dim M = 2n$, and μ is a choice of moment map associated to the action. We say two symplectic toric manifolds $(M_1, \omega_1, \mathbb{T}_1^n, \mu_1)$ and $(M_2, \omega_2, \mathbb{T}_2^n, \mu_2)$ are *equivalent* if there is an isomorphism $\lambda : \mathbb{T}_1^n \to \mathbb{T}_2^n$ and a λ-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that $\mu_1 = \mu_2 \circ \varphi$.

Two examples are the following:

(1) Let S^1 act on S^2 via rotations around the z-axis. Then, the action is hamiltonian with moment map $\mu(x, y, z) = z$.
(2) Let $\mathbb{T}^n \cong (S^1)^n$ act on $\mathbb{C}^n$ by rotating each factor. That is, our action is defined by $(e^{it_1}, \ldots, e^{it_n}) \cdot (z_1, \ldots, z_n) = (e^{it_1} z_1, \ldots, e^{it_n} z_n)$ and has moment map $\mu(z_1, \ldots, z_n) = -\frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$.

The set $\mu(M)$ is a polytope. In the case where $(M, \omega, \mathbb{T}^n, \mu)$ is a symplectic toric manifold, $\mu(M)$ is a special polytope called a Delzant polytope.

Definition 2. A polytope $\Delta \subseteq \mathbb{R}^n$ is a *Delzant polytope* if the following three conditions hold:

(a) Simplicity - that is, at every vertex p there are precisely n edges $u_1, \ldots, u_n$ at p.
(b) Rationality - that is, at every vertex p the edges u_i can be written in the form $p + tv_i$ where $v_i \in \mathbb{Z}^n$ and t is in some closed interval in $\mathbb{R}$.
(c) Smoothness - that is, at every vertex p the $v_1, \ldots, v_n$ from (b) form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

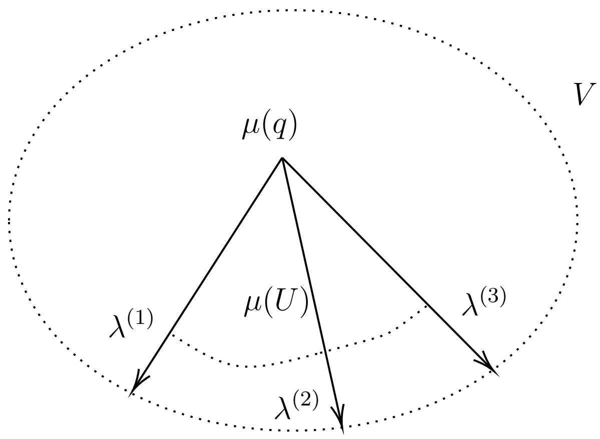
We know that the verticies of our moment polytope $\mu(M)$ come from fixed points but the opposite is not always true for a general Hamiltonian torus action on M. However, in the case of a symplectic toric manifold we can actually say that every fixed point is actually mapped to a vertex.

We sketch a proof here. We borrow the following result which allows us to write our moment map in a nice form locally near a fixed point.

Theorem 3. Let $(M^{2n}, \omega, \mathbb{T}^m, \mu)$ be a Hamiltonian $\mathbb{T}^m$-space, and let q be a fixed point. There exists a chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at q and elements $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^m$ such that

$$\mu|_U = \mu(q) - \frac{1}{2}\sum_{k=1}^{n} \lambda^{(k)}(x_k^2 + y_k^2).$$

Now, let q be a fixed point. If $\mu(q)$ is an interior point of $\mu(M)$, then there is some open set $V \subseteq \mu(M)$ surrounding $\mu(q)$. Visually we have the following picture:



We have that $\mu : M \to \mu(M)$ is an open map and so $\mu(U) \subseteq V$ is open. However, visually this is clearly not the case and so $\mu(q)$ must be a vertex. So, we have a bijection relating fixed points of the action to verticies in the moment polytope. □

One immediate observation is that the $\lambda^{(k)}$ at a vertex $\mu(q)$ are the elements v_k in part (b) of Definition 2. So, the $\lambda^{(k)}$ define the edges of our polytope.

The bijection between fixed points of the action and verticies of $\mu(M)$ is very nice. And, in fact there is an exremely nice result which states that symplectic toric manifolds are classified by their moment polytopes. More precisely there is a bijective map

$$\{\text{symplectic toric manifolds}\} \to \{\text{Delzant polytopes}\}$$
$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \mapsto \mu(M).$$

The proof involves taking a Delzant polytope and constructing a particular symplectic reduction of Example (2).

3. Morse Theory

In order to compute the homology groups of the symplectic toric manifolds, we need to introduce some fundamental results from Morse theory.

To begin, we define a what a morse function is.

Definition 4. A *Morse function* $f : M \to \mathbb{R}$ is a function where the critical points of f are nondegenerate. That is, if $df_q = 0$ then $\det H_q \neq 0$ where H_1 is the Hessian matrix of f at q. The index of a bilinear map $H_q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is dimension of the largest subspace of

$\mathbb{R}^m$ upon which H_q is negative definite, and if q is a nondegenerate critical point of f we call the index of the Hessian H_q the *index of f at q*.

At a nondegerate critical point q, there are coordinates $(U, x_1, \ldots, x_m)$ such that

$$f|_U = f(q) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_m^2$$

where λ is the index of f at q. Theorem 3 tells us that the moment map for a hamiltonian $\mathbb{T}^m$-space at a fixed point looks very similar to a morse function but maps to a space of several dimensions.

Now we state a theorem which relate Morse functions to CW-complexes. The theorem will bridge our discussion about moment maps with homology.

Theorem 5. Let $f : M \to \mathbb{R}$ be a Morse function, and write $M^a = f^{-1}((-\infty, a])$. Then, the following hold:

(a) Let $a < b$ and suppose $f^{-1}([a, b])$ has no critical points of f. Then, M^a is diffeomorphic to M^b, M^a is a deformation retract of M^b, and $M^a \hookrightarrow M^b$ is a homotopy equivalence.
(b) If q is a nondegenerate critical point with index λ and $f(q) = c$ with $f^{-1}([c-\epsilon, c+\epsilon])$ compact and contains no critical points of f besides q then $M^{c+\epsilon}$ has the same homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.
(c) If M^a is compact for all a, then M has the homotopy type of a CW-complex with one cell of each dimension λ for each critical point of index λ.

We will only use part (c) of Theorem 5.

4. Homology

We are now ready to talk about the homology groups of M. First, let us pick some $X \in \mathbb{R}^n$ such that the components of X are independent over $\mathbb{Q}$. Let $X^\sharp$ be the vector field on $\mathbb{T}^n$ defined by X. Because the components are independent over $\mathbb{Q}$, we have that the one-parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\}$ is dense in $\mathbb{T}^n$. The zeroes of $X^\sharp$ are precisely the fixed points of the action, and since our action is Hamiltonian we have that $d\mu^X = \iota_{X^\sharp}\omega$ and so $d\mu^X = 0$ when $X^\sharp = 0$. So, the critical points of $d\mu^X$ are the zeroes of $X^\sharp$ which are the fixed points of the action which are in bijection with the verticies of the Delzant polytope $\mu(M)$.

Now let q be a critical point of $d\mu^X$. We know that $\mu^X = \langle \mu, X \rangle$ and by Theorem 3 and linearity of the inner product, we have that locally around some open set U containing q, there are coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that

$$\mu^X|_U = \mu^X(q) - \frac{1}{2}\sum_{k=1}^{n} \langle \lambda^{(k)}, X \rangle (x_k^2 + y_k^2).$$

Since X has components which are independent over $\mathbb{Q}$, $\langle \lambda^{(k)}, X\rangle \neq 0$ for all $k \in \{1, \ldots, n\}$. The Hessian H_q of μ^X at q is

$$\begin{pmatrix} \langle \lambda^{(1)}, X\rangle & & & & & \\ & \ddots & & & & \\ & & \langle \lambda^{(n)}, X\rangle & & & \\ & & & \langle \lambda^{(1)}, X\rangle & & \\ & & & & \ddots & \\ & & & & & \langle \lambda^{(n)}, X\rangle \end{pmatrix}$$

and is invertible. So, μ^X is actually a Morse function with index $2\left|\{k : \langle \lambda^{(k)}, X\rangle > 0\}\right|$. In particular there are no critical points with odd index. Since M is assumed to be compact, by part (c) of Theorem 5 our CW-complex tells us that our chain groups C_k in our chain complex

$$\ldots \longrightarrow C_4 \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

are free abelian groups of rank b_k where b_k is the number of critical points with index k. There are no critical points with odd index, so $C_{2k+1} = 0$. And, $b_{2k} = 2\left|\{k : \langle \lambda^{(k)}, X\rangle > 0\}\right|$ so $C_{2k} = \mathbb{Z}^{b_{2k}}$. Then our chain complex is of the form

$$\ldots \longrightarrow \mathbb{Z}^{b_4} \longrightarrow 0 \longrightarrow \mathbb{Z}^{b_2} \longrightarrow 0 \longrightarrow \mathbb{Z}^{b_0} \longrightarrow 0$$

and so

$$H_k(M) = \begin{cases} 0 & k = 2m+1, \\ \mathbb{Z}^{2\left|\{m : \langle \lambda^{(m)}, X\rangle > 0\}\right|} & k = 2m. \end{cases}$$

We note that the condition $\langle \lambda^{(m)}, X\rangle > 0$ is equivalent to $\lambda^{(m)}$ lying in the upper half-space defined by X. So, geometrically $\langle \lambda^{(m)}, X\rangle > 0$ means that $\lambda^{(m)}$ "points upwards" with respect to X. So for each k we can just count how many verticies have edges pointing upwards with respect to X and multiply by two. That will give us the rank of our homology groups, which we now know to be free abelian groups.

References

[1] Cannas da Silva, A., *Symplectic Toric Manifolds* (2001)