Character Varieties
Rigid Local Systems Seminar - February 17, 2023

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February 16, 2023

I want to talk about how we can study representations $\Gamma \to G$ by viewing them as parametrized by a geometric space, and how we can compute tangent spaces of this geometric space in terms of group cohomology. Finally, I want to use this to compute the dimension of the tangent space when $\Gamma = \pi_1(S_g)$ where S_g is a closed orientable surface of genus g.

1 The Representation Variety and the Character Variety

Let Γ be a finitely generated group and G a linear algebraic group (in most cases, we care about $G = GL_n$). We call $\operatorname{Hom}(\Gamma, G)$ the *representation variety* of Γ. In what sense is this actually a geometric space? In the case of the free group on the generators $e_1, \ldots, e_n$, we have that every element of $\operatorname{Hom}(F_n, G)$ is determined by where e_i is sent. So, $\operatorname{Hom}(F_n, G) = G^n$. Since G is an algebraic group, $\operatorname{Hom}(\Gamma, G)$ is also an algebraic group. In the case where Γ is the quotient of some free group with relations $r_1, \ldots, r_m$, then we are constrained by the equations $r_1 = \cdots = r_m = 1$.

Example 1. Let's look at the character variety $\operatorname{Hom}(\mathbb{Z}^2, SL_2(\mathbb{C}))$. To do so, consider $\operatorname{Hom}(F_2, SL_2(\mathbb{C}))$. Then, $\rho \in \operatorname{Hom}(F_2, SL_2(\mathbb{C}))$ is determined by $\rho((1,0))$ and $\rho((0,1))$. So,

$$\rho((1,0)) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$
$$\rho((0,1)) = \begin{pmatrix} x_5 & x_6 \\ x_7 & x_8 \end{pmatrix}$$

and so our $\operatorname{Hom}(F_2, SL_2(\mathbb{C}))$ is a subvariety of $\mathbb{A}^8_{\mathbb{C}}$ subject to the condition that $\det \rho((1,0)) = \det \rho((0,1)) = 1$. We get $\operatorname{Hom}(F_2, SL_2(\mathbb{C})) = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_8]/(x_1x_4 - x_2x_3 - 1, x_5x_8 - x_6x_7 - 1)$. Then, $F_2/[F_2, F_2] \cong \mathbb{Z}^2$ so in order to compute $\operatorname{Hom}(\mathbb{Z}^2, SL_2(\mathbb{C}))$ we just need to add additional constraints given by looking at

$$\begin{pmatrix} x_1x_5 + x_3x_6 & x_2x_6 + x_4x_8 \\ x_1x_7 + x_3x_8 & x_2x_7 + x_4x_8 \end{pmatrix} = \begin{pmatrix} x_5 & x_6 \\ x_7 & x_8 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} x_5 & x_6 \\ x_7 & x_8 \end{pmatrix} = \begin{pmatrix} x_1x_5 + x_2x_7 & x_1x_6 + x_2x_8 \\ x_3x_5 + x_4x_7 & x_3x_6 + x_4x_8 \end{pmatrix}.$$

Then, this gives us the equations

$$\begin{aligned} x_3x_6 - x_2x_7 &= 0 \\ x_2x_6 + x_4x_8 - x_1x_6 - x_2x_8 &= 0 \\ x_1x_7 + x_3x_8 - x_3x_5 - x_4x_7 &= 0 \\ x_2x_7 - x_3x_6 &= 0 \end{aligned}$$

So, $\mathrm{Hom}(\mathbb{Z}^2, SL_2(\mathbb{C}))$ is the closed subscheme of $\mathbb{A}^8_{\mathbb{C}}$ cut out by

$$\begin{aligned} x_1x_4 - x_2x_3 - 1 &= 0 \\ x_5x_8 - x_6x_7 - 1 &= 0 \\ x_2x_7 - x_3x_6 &= 0 \\ x_2x_6 + x_4x_8 - x_1x_6 - x_2x_8 &= 0 \\ x_1x_7 + x_3x_8 - x_3x_5 - x_4x_7 &= 0 \\ x_2x_7 - x_3x_6 &= 0 \end{aligned}$$

In fact, this is an irreducible subscheme but we will not prove it. If we want to replace $SL_2(\mathbb{C})$ with $GL_2(\mathbb{C})$ in the example above, we replace the $x_1x_4 - x_2x_3 - 1 = x_5x_8 - x_6x_7 - 1 = 0$ condition with $x_1x_4 - x_2x_3 \neq 0$ and $x_5x_8 - x_6x_7 \neq 0$.

Definition 4. Let $\mathrm{Hom}(\Gamma, G)$ be a representation variety, and let G act on $\mathrm{Hom}(\Gamma, G)$ act by conjugation. Then, the quotient $\chi_G(\Gamma) := \mathrm{Hom}(\Gamma, G)//G$ is the *character variety*.

Example 5. Fricke's Theorem states that $\chi_{SL_2}(F_2) \cong \mathbb{A}^3_{\mathbb{C}}$. The map is given as follows: a point in $\chi_{SL_2}(F_2)$ is given by a pair of conjugacy classes of matrices in SL_2. So, we have a point $([A], [B])$ and the map is given by $([A], [B]) \mapsto (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB))$. I'm not going to prove it, but you can find the proof in "*Trace coordinates on Fricke spaces of some simple hyperbolic surfaces*" by William Goldman on arXiv.

2 Cohomology of the adjoint representation, and tangent spaces

In this section, we want to talk about how to describe $T_\rho\chi_G(\Gamma)$. Specifically, we discuss an isomorphism $T_\rho\chi_G(\Gamma) \cong H^1(\Gamma, Ad\,\rho)$ where the right-hand-side is group cohomology twisted by the adjoint representation. First, we recall the definition of group cohomology.

Definition 6. We give the definition of $H^1(\Gamma, Ad\,\rho)$. Let $f : \Gamma \to \mathfrak{g}$ be any function such that for all $g_1, g_2 \in \Gamma$

$$f(g_1g_2) = Ad\,\rho(g_1) \cdot f(g_2) + f(g_1).$$

Let $Z^1(\Gamma, Ad\,\rho)$ be the collection of all f functions which satisfy this. This is just the condition that $f \in \ker d^2$, where $d^2 : C^1(\Gamma, Ad\,\rho) \to C^2(\Gamma, Ad\,\rho)$ is the co-boundary map between the first and second co-chains in the co-chain complex defining group cohomology. Then, let $B^1(\Gamma, Ad\,\rho) = \{Ad\,\rho(g) \cdot f(e) - f(e) : f : \{e\} \to \mathfrak{g} \text{ is a function}\}$. This is the image of the coboundary map d^1. Then, $H^1(\Gamma, Ad\,\rho) := Z^1(\Gamma, Ad\,\rho)/B^1(\Gamma, Ad\,\rho)$ is the first cohomology group.

Definition 7. A $\mathbb{C}$-algebra $R(\Gamma, G)$ is a *universal representation of* Γ *into* G, and $\rho_U : \Gamma \to G(R(\Gamma, G))$ is a *universal representation* if for every $\mathbb{C}$-algebra homomorphism A and every representation $\rho : \Gamma \to G(A)$, there is a $\mathbb{C}$-algebra homomorphism $f : R(\Gamma, G) \to A$ inducing $G(f) : G(R(\Gamma, G)) \to G(A)$ such that $\rho = G(f)\rho_U$. As a diagram,

$$\begin{array}{ccc} & & G(R(\Gamma, G)) \\ & \overset{\rho_U}{\nearrow} & \Big\downarrow{\scriptstyle G(f)} \\ \Gamma & \xrightarrow[\rho]{} & G(A) \end{array}$$

the dashed line above exists.

This exists, and is given by the following: $\mathbb{C}[G]$ as $\mathbb{C}[GL_n] = (\mathbb{C}[d, x_{ij}, 1 \leq i, j \leq n]/(d \cdot \det(x_{ij}) - 1))/I = \mathbb{C}[d, x_{ij}]/I$ (this gives us the nonzero determinant condition), if $\Gamma = F_N = \langle \gamma_1, \ldots, \gamma_N \rangle$ a free group we just get

$$R(F_N, G) = \mathbb{C}[d_1, x_{1ij}]/I \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[d_N, x_{Nij}]/I$$

and $\rho_U(\gamma_t) = x_{tij}$ satisfies the properties. Then, if $\Gamma = F_N/H$ we get a quotient of $R(F_N, G)$ by respecting the relations given by H.

Let ρ be a point with residue field $\mathbb{C}$ (i.e., a closed point in our representation variety. Let $r_{\mathfrak{m}} : \mathbb{C}[\mathrm{Hom}(\Gamma, G)] \to \mathbb{C}[\mathrm{Hom}(\Gamma, G)]/\mathfrak{m} \cong \mathbb{C}$ be the projection given by inclusion of the point $\rho \hookrightarrow \mathrm{Hom}(\Gamma, G)$. Then, the tangent space at ρ is given by maps

$$\mathbb{C}[\mathrm{Hom}(\Gamma, G)] \xrightarrow{v} \mathbb{C}[\epsilon]/(\epsilon^2) \xrightarrow{\pi:\epsilon\mapsto 0} \mathbb{C}$$

such that the composition $\mathbb{C}[\mathrm{Hom}(\Gamma, G)] \to \mathbb{C}$ is given by $r_{\mathfrak{m}}$. Intuitively, the algebra morphism v is a tangent vector, and the condition that the composition is given by $r_{\mathfrak{m}}$ just says that if our vector is 0, we should get the actual point on our space.

Every such homomorphism (and therefore tangent vector at ρ) given above is of the form $r_{\mathfrak{m}} + \epsilon v$, we get that $G(r_{\mathfrak{m}} + \epsilon v) \circ \rho_U(\gamma) \in G(\mathbb{C}[\epsilon]/(\epsilon^2))$. Then for $\gamma \in \Gamma$, we define a map $\sigma : \Gamma \to \mathfrak{g}$ given by

$$\sigma(\gamma) := \frac{(G(r_{\mathfrak{m}} + \epsilon v)\rho_U(\gamma))(G(r_{\mathfrak{m}})\rho_U(\gamma))^{-1} - I}{\epsilon} \in \mathfrak{g}.$$

Here, our A is given by $\mathbb{C}[\epsilon]/(\epsilon^2)$, this definition mirrors that of the derivative. One key property this map σ satisfies is that

$$\sigma(g_1 g_2) = Ad\,(\pi \circ \rho(g_1)) \cdot \sigma(g_2) + \sigma(g_1).$$

Note that here, $\pi \circ \rho = \rho$, since π sends a tangent vector in $T_\rho \mathrm{Hom}(\Gamma, G)$ to ρ. This is precisely the condition that $\sigma \in \ker d^2 = Z^1(\Gamma, Ad\,\rho)$.

So, the map $\Psi_\rho(t) = \sigma$ is a linear map $T_\rho \mathrm{Hom}(\Gamma, G) \to Z^1(\Gamma, Ad\,\rho)$. This is in fact an isomorphism with inverse $\Phi_\rho : \sigma \mapsto (\gamma \mapsto (I + \sigma(\gamma)\epsilon)\rho(\gamma))$ which is a homomorphism $\Gamma \to G(\mathbb{C}[\epsilon]/(\epsilon^2))$, i.e., an element of $T_\rho \mathrm{Hom}(\Gamma, G)$.

The important theorem that we can now state is the following:

Theorem 8. Suppose $\rho : \Gamma \to G = GL_n$ is a semisimple representation. Then, the isomorphism $\Phi_\rho : Z^1(\Gamma, Ad\,\rho) \to T_\rho \mathrm{Hom}(\Gamma, G)$ induces a linear map $\widetilde{\Phi}_\rho : H^1(\Gamma, Ad\,\rho) \to T_\rho \chi_G(\Gamma)$. And, when ρ is an irreducible representation, $\widetilde{\Phi}_\rho$ is actually an isomorphism.

Proof. (Sketch) We give some reasoning why we get an isomorphism of tangent spaces. Let O_ρ be the orbit of ρ under the action by G. If we let $f_\rho : G \to \mathrm{Hom}(\Gamma, G)$ be given by $g \mapsto g\rho g^{-1}$, then $\mathrm{Im}(f_\rho) = O_\rho$. So, $T_\rho O_\rho = \mathrm{Im}(df_\rho) = \{(df_\rho)(I + \epsilon A) : A \in G, \epsilon^2 = 0\}$.

Every tangent vector in $T_\rho \mathrm{Hom}(\Gamma, G)$ comes from a map $I + \epsilon A$ where $A \in G$, so every vector v is given by the map

$$\gamma \mapsto (I + \epsilon A)\rho(\gamma)(I + \epsilon A)^{-1}$$

since our map f_ρ is the conjugation action. So,

$$\begin{aligned}(I + \epsilon A)\rho(\gamma)(I + \epsilon A)^{-1} &= (I + (A - \rho(\gamma)A\rho(\gamma)^{-1})\epsilon)\rho(\gamma) \\ &= (I + \tau\epsilon)\rho(\gamma)\end{aligned}$$

where $\tau = A - \rho(\gamma)A\rho(\gamma)^{-1} = A - Ad\,\rho(\gamma) \cdot A$. Note that here, τ is precisely an element of $B^1(\Gamma, Ad\,\rho)$! So, we get an isomorphism $\widetilde{\Phi}_\rho : H^1(\Gamma, Ad\,\rho) \to T_\rho \mathrm{Hom}(\Gamma, G)/T_\rho O_\rho$.

In the case where ρ is irreducible, we have an isomorphism $T_\rho \mathrm{Hom}(\Gamma, G)/T_\rho O_\rho \to T_\rho \chi_G(\Gamma)$ so we get the isomorphism as desired. (Étale slice). □

Example 9. How can we compute $\dim T_\rho \chi_G(\Gamma)$ where $\Gamma = \pi_1(S_g)$ and S_g is a closed orientable surface of genus g and $G = SL_n$? Here, $\dim SL_n = n^2 - 1$. We have that

$$(2 - 2g) \dim SL_n = \dim H^0(\Gamma, Ad\,\rho) - \dim H^1(\Gamma, Ad\,\rho) + \dim H^2(\Gamma, Ad\,\rho).$$

Then, $H^0(\Gamma, Ad\,\rho) = \mathfrak{sl}_n^{\pi(S_g)}$ (the elements fixed by $\pi(S_g)$). This is because of the long exact sequence on group cohomology. Since ρ is irreducible, $\dim H^0 = 0$. Since H^2 is dual to H^0 since we are on a surface, we get that

$$\dim T_\rho \chi_G(\pi(S_g)) = \dim H^1(\Gamma, Ad\,\rho) = (2g - 2) \dim SL_n = (2g - 2)(n^2 - 1).$$

3 Fixed local monodromy

(Only if have time!) Let $j : X = \mathbb{P}^1 \setminus \{p_1, \dots, p_{n+1}\} \hookrightarrow \mathbb{P}^1$ be the standard inclusion, and $\mathfrak{g}$ a local system on X. Then we can look at the pushforward $j_*\mathfrak{g}$ which is a local system on $\mathbb{P}^1$. Then, we get a short exact sequence of sheaves

$$0 \longrightarrow \mathfrak{g} \longrightarrow j_*\mathfrak{g} \longrightarrow \oplus_i \mathfrak{g}^{\gamma_i} \longrightarrow 0$$

where $\oplus_i \mathfrak{g}^{\gamma_i}$ is a direct sum of skyscraper sheaves supported at the p_i. Here, our stalk is just the same as the stalk of $\mathfrak{g}$ at p_i.

Since we get a short exact sequence of sheaves on $\mathbb{P}^1$, we get a long exact sequence in sheaf cohomology. So, we get

$$0 \longrightarrow H^0(\mathbb{P}^1, \mathfrak{g}) \xrightarrow{f_0} H^0(\mathbb{P}^1, j_*\mathfrak{g}) \xrightarrow{g_0} H^0(\mathbb{P}^1, \oplus_i \mathfrak{g}^{\gamma_i}) \xrightarrow{\delta} H^1(\mathbb{P}^1, \mathfrak{g}) \xrightarrow{f_1} H^1(\mathbb{P}^1, j_*\mathfrak{g}) \xrightarrow{g_1} H^1(\mathbb{P}^1, \oplus_i \mathfrak{g}^{\gamma_i}) = 0 \longrightarrow$$

Since $\oplus_i \mathfrak{g}^{\gamma_i}$ is a skyscraper sheaf the higher cohomology groups vanish. Then, the dimension of the tangent space at an irreducible representation is given by $\dim H^1(\mathbb{P}^1, j_*\mathfrak{g})$ and we can compute this number. That is,

$$\begin{aligned}
\dim H^1(\mathbb{P}^1, j_*\mathfrak{g}) &= \dim \ker g_1 = \dim \operatorname{Im} f_1 = \dim H^1(\mathbb{P}^1, \mathfrak{g}) - \dim \operatorname{Im} \delta \\
&= \dim H^1(\mathbb{P}^1, \mathfrak{g}) - (\dim H^0(\mathbb{P}^1, \oplus_i \mathfrak{g}^{\gamma_i}) - \ker \delta) \\
&= \dim H^1(\mathbb{P}^1, \mathfrak{g}) - (\dim H^0(\mathbb{P}^1, \oplus_i \mathfrak{g}^{\gamma_i}) - \operatorname{Im} g_0) \\
&= \dim H^1(\mathbb{P}^1, \mathfrak{g}) - \dim H^0(\mathbb{P}^1, \oplus_i \mathfrak{g}^{\gamma_i}) + \dim H^0(\mathbb{P}^1, j_*\mathfrak{g}) - \ker g_0 \\
&= \dim H^1(\mathbb{P}^1, \mathfrak{g}) - \dim H^0(\mathbb{P}^1, \oplus_i \mathfrak{g}^{\gamma_i}) + \dim H^0(\mathbb{P}^1, j_*\mathfrak{g}) - \operatorname{Im} f_0 \\
&= \dim H^1(\mathbb{P}^1, \mathfrak{g}) - \dim H^0(\mathbb{P}^1, \oplus_i \mathfrak{g}^{\gamma_i}) + \dim H^0(\mathbb{P}^1, j_*\mathfrak{g}) - \dim H^0(\mathbb{P}^1, \mathfrak{g}).
\end{aligned}$$