Gromov-Witten Invariants and Curve Counting Talk Notes

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Abstract

We discuss some aspects of Gromov-Witten theory, specifically when the Gromov-Witten invariants are enumerative.

1 Introduction

We begin with a simple question: how do we count rational curves in a given space? We begin with an example.

Question 1. There are infinitely many rational curves in $\mathbb{P}^n$.

This isn't a very interesting question, since we want an actual finite number.

Question 2. How many rational curves pass through k points in $\mathbb{P}^2$?

Here, we have specified conditions on our curves to try and get a finite number. Unfortunately, this still results in an infinite number. For example, if $k = 2$, we still get an infinite number because we can draw an infinite family of quadratics which pass through two points. But if restrict two a degree 1 rational curve (i.e., a line), we get a unique line passing through two points.

So, perhaps we should ask the following:

Question 3. How many degree d curves in $\mathbb{P}^2$ through k points?

This is the first question where we might get a finite answer (say, when $d = 1$ and $k = 2$). When $d = 2$, a rational conic is specified by 5 points. Consider $p_1, \dots, p_5 \in \mathbb{P}^2$. We can assume they are away from the line at infinity $V(z)$ and hence write them as $p_j = [x_j, y_j, 1]$. Then, the polynomial $f(X, Y)$ given by

$$f(X,Y) = \det \begin{pmatrix} 1 & X & Y & X^2 & Y^2 & XY \\ 1 & x_1 & y_1 & x_1^2 & y_1^2 & x_1y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_5 & y_5 & x_5^2 & y_5^2 & x_5y_5 \end{pmatrix}$$

gives us the (unique!) rational curve passing through these 5 points. And, these 5 points are necessary since given just 4 points, we can generically pick a fifth point and do the above construction to get a degree 2 curve passing through these 5 points.

Notice that for $d = 1, 2$ the number of points needed to specify a finite number of these curves is $3d - 1$. This pattern actually continues.

Proposition 4. Let $X_{d,k}$ be the space of rational degree d curves passing through k generically chosen points in $\mathbb{P}^2$. If $k \neq 3d - 1$, then $X_{d,k}$ is either empty or infinite.

Proof. A rational degree d curve is the image of a degree d morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$. Such a morphism is given by three non-simultaneously vanishing sections $s_0, s_1, s_2 \in H^0(\mathscr{O}_{\mathbb{P}^1}(d))$. Then, $\dim \mathscr{O}_{\mathbb{P}^1}(d) = d+1$ so we seek $Z \subseteq H^0(\mathscr{O}_{\mathbb{P}^1}(d))^3$ where Z is the subvariety of $H^0(\mathscr{O}_{\mathbb{P}^1}(d))^3$ consisting of sections which do not simultaneously vanish. Non-vanishing is an open condition, so $\dim Z = 3d+3$. Note that simultaneously scaling each s_i by some $\lambda \in \mathbb{C}^\times$ yields the same morphism, so $\varphi \in Z/\mathbb{C}^\times$ and $\dim Z/\mathbb{C}^\times = 3d+3-1 = 3d-2$.

Because we are only concerned with the image of φ, we can quotient by the action of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ to get $X_{d,0} = (Z/\mathbb{C}^\times)/(\mathrm{PGL}_2)$ and $\dim X_{d,0} = 3d+2-\dim \mathrm{PGL}_2 = 3d+2-3 = 3d-1$. Then, if we require our curve to pass through a generically chosen point p, this cuts down our dimension by 1. Hence, $\dim X_{d,k} = 3d-1-k$. To get a finite set, we require $\dim X_{d,k} = 0 = 3d-1-k$ and so $k = 3d-1$. □

So, this question is only interesting when $k = 3d-1$.

To formulate this question more generally, we seek a cohomological description.

2 Cohomological Formulations

Definition 5. A genus g, n-marked pre-stable curve consists of the data $(C, x_1, \ldots, x_n)$ where

1. C is a curve of arithmetic genus g, i.e., $\chi(\mathscr{O}_C) = 1-g$. (This differs from the geometric genus defined as $\dim H^{1,0}(C, \Omega_C)$.

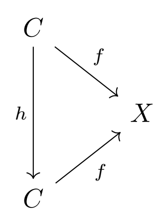
2. x_i are smooth points of C.

Let X be a smooth projective variety. Then a genus g, n-marked stable map into X consists of

1. A genus g, pre-stable curve $(C, x_1, \ldots, x_n)$.

2. A map $f : C \to X$ with only finitely many automorphisms

where an automorphism h is a map making the diagram



commute.

We let $\overline{\mathcal{M}}_{g,n}(X) = \{f : (C, x_1, \ldots, x_n) \to X : f \text{ is stable}\}$. If $\beta \in H_2(X)$ is a homology class, then we say $\overline{\mathcal{M}}_{g,n}(X, \beta) = \{f : (C, x_1, \ldots, x_n) \to X : f \text{ is stable}, f_*[C] = \beta\}$.

Example 6. Consider $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 1)$, where $1 \in \mathbb{Z} \cong H_2(\mathbb{P}^2)$. Then, this is the collection of maps $f : \mathbb{P}^1 \to \mathbb{P}^2$ such that $f_*[\mathbb{P}^1] = 1$. But this is just the collection of lines in $\mathbb{P}^2$ which is $(\mathbb{P}^2)^\vee \cong \mathbb{P}^2$.

These spaces are not all quite so nice, however. The space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2)$ should be all the conics in $\mathbb{P}^2$, but this isn't. This is because not all maps $f : C \to \mathbb{P}^2$ will be smooth. Due to this, there are multiple components of this space of different dimensions.

From now on, we make the assumption that $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is smooth, compact, and all components have the same dimension. We define

$$\Gamma = \mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X) \to X$$

given by $f : (C, x_1, \ldots, x_n) \to X \mapsto f(x_i)$.

Let $V_1, \ldots, V_n$ be subvarieties of X. Then, we get cohomology classes $\gamma_j \in H^k(X)$. Then, $\mathrm{ev}_i^*\gamma_i$ is a cohomology class, and the Poincaré dual gives us the collection of maps $f : C \to X$ such that $f(x_i) \in V_i$. We know that the cup product in cohomology is dual to intersection in homology, the class Γ represents the maps $f : C \to X$ such that $f(x_i) \in V_i$ for all $1 \leq i \leq n$. Because the x_i vary over C, the cohomology class is just given by morphisms $f : C \to X$ such that $f(C)$ intersects V_i for all $1 \leq i \leq n$. If this is a top form in $H^\bullet(\overline{\mathcal{M}}_{g,n}(X, \beta))$ then pairing it with the top homology class $[\overline{\mathcal{M}}_{g,n}(X, \beta)]$ gives us the number of curves satisfying our conditions. That is, we want to consider

$$\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \Gamma.$$

Definition 7. Let $\gamma_1, \ldots, \gamma_n \in H^*(X)$. We define the Gromov-Witten invariant

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta}^X = \int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \mathrm{ev}_1^*\gamma_1 \smile \cdots \smile \mathrm{ev}_n^*\gamma_n.$$

Example 8. The Gromov-Witten invariant $\langle p_1, \ldots, p_{3d-1} \rangle_{0,d}^{\mathbb{P}^2}$ is precisely the number of degree d rational curves in $\mathbb{P}^2$ passing through $3d - 1$ points.

So, we have given a cohomological formulation of this curve counting business, and we can ask this question whenever we have cohomology classes which give us a top form.

3 Properties of Gromov-Witten Invariants

Definition 9. Let $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ be the morphism $(f : (C, x_1, \ldots, x_{n+1}) \to X) \mapsto (f : C(x_1, \ldots, x_n) \to X)$. This is called a forgetful morphism.

We have the map $\pi : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$ where $(f : (C, x_1, \ldots, x_n) \to X) \mapsto (C, x_1, \ldots, x_n)$.

These maps exist as long as the actual target moduli space exists. The reason it doesn't always exist is because when sending $(f : (C, x_1, \ldots, x_{n+1}) \to X) \to (f : (C, x_1, \ldots, x_n) \to X)$ or $(f : (C, x_1, \ldots, x_n) \to X) \to (C, x_1, \ldots, x_n)$, the resulting curve need not be stable (having finitely many automorphisms). But this can be resolved by a process called stabilaziation (and we won't worry about this). If $n > 2 - 2g$, then our curve is always stable. For example, when $g = 0$, an automorphism sending 3 points of $\mathbb{P}^1$ must be a permutation of those points, and hence there are at most S_3 of them.

The Gromov-Witten invariants satisfy a collection of axioms:

Proposition 10.

1. We have the equality

$$\langle \gamma_1, \ldots, \gamma_n, [X]^\vee \rangle_{g,\beta}^X = \langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta}^X.$$

   This is saying that $f(x_{n+1}) \in X$ is an empty condition. Cohomologically, this is saying that

$$\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \mathrm{ev}_1^*\gamma_1 \smile \cdots \smile \mathrm{ev}_n^*\gamma_n \smile [X]^\vee = \int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \mathrm{ev}_1^*\gamma_1 \smile \cdots \smile \mathrm{ev}_n^*\gamma_n$$

   and since $[X]^\vee \in H^0(X) \cong \mathbb{Z}$ is the fundamental class, this is just multiplication by 1.

2. If $\gamma_n \in H^2(X, \mathbb{Q})$, then

$$\langle \gamma_1, \ldots, \gamma_{n-1}, \gamma_n \rangle_{g,\beta}^X = \left(\int_\beta \gamma_n \right) \langle \gamma_1, \ldots, \gamma_{n-1} \rangle_{g,\beta}^X.$$

   This is because the a curve intersects a divisor at $\int_\beta \gamma_n$-many points.

3. If $\beta = 0$, then

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,0}^X = \begin{cases} \int_X \gamma_1 \smile \gamma_2 \smile \gamma_3 & n = 3 \\ 0 & \text{otherwise.} \end{cases}$$

If S is a complex surface, then the divisors on S are codimension 1 subvarieties. So, we can use Axiom 2 to reduce to the case where we are computing $\langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta}^S$ where $\gamma_1, \ldots, \gamma_n$ come from points. For $S = \mathbb{P}^2$, the following result of Kontsevich answers our question completely.

Theorem 11 - Kontsevich Recursion. Let N_d be the number of degree d curves passing through $3d-1$ points. The following recursive formula computes N_d:

$$N_d = \sum_{d_1+d_2=d} \left(d_1^2 d_2^2 N_{d_1} N_{d_2} \binom{3d-4}{3d_1-2} - d_1^3 d_2 N_{d_1} N_{d_2} \binom{3d-4}{3d_1-1} \right).$$

The first few numbers are given by

d	$3d-1$	N_d
1	2	1
2	5	1
3	8	12
4	11	620
5	14	87304
6	17	2631297
7	20	14616808192
8	23	13525751027392
9	26	19385778269260800
10	29	40739017561997799680
$\vdots$	$\vdots$	$\vdots$

For E a curve, the divisors are given by just points. So, using Axiom 2 we know that $\langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta}^E$ can be computed by just computing the empty bracket $\langle \rangle_{g,\beta}^E$.

4 References

"An Introduction to Gromov-Witten Theory" by Simon Rose, https://arxiv.org/abs/1407.1260