Why an Engineer Can Draw Better Than You

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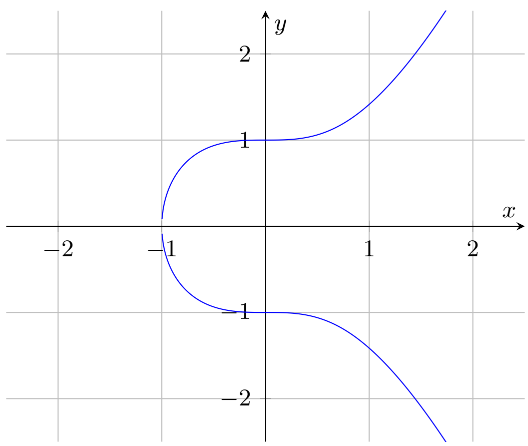
Abstract

One of the most fundamental ways to understand what a mathematical object is to figure out what it looks like. This is most easily done when one can actually draw the object. However, drawing an object accurately in real life is actually quite a difficult task! We discuss how to draw polynomial curves precisely, and why it's probably not actually worth doing so.

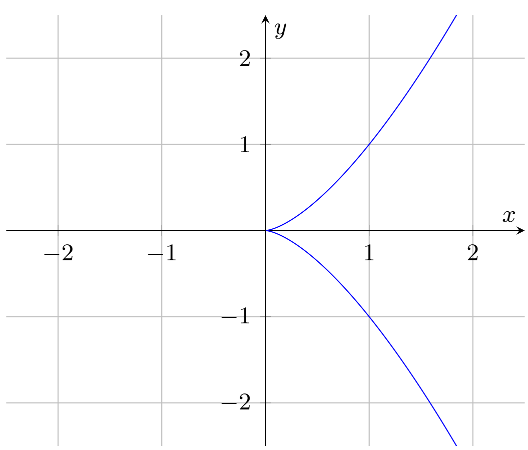
1 Drawing

Drawing pictures can provide a great understanding of what objects look like. Polynomials are examples of things which show up everywhere in mathematics. But this is very difficult to draw in general! By drawing polynomials, we mean drawing the zero locus (i.e., $f(x, y) = 0$). (Note that if we want to draw the graph of a polynomial $y = f(x)$, then this is just the zero set of the polynomial $g(x, y) = y - f(x)$.) Even similar-looking polynomials can have widly different shapes.

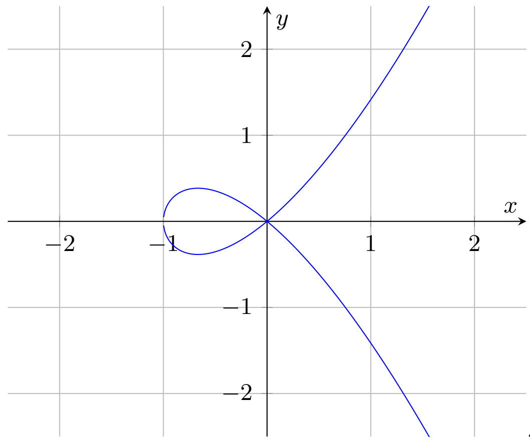
Example 0.3. 1. $f(x, y) = y^2 - x^3 - 1 = 0$.



2. $f(x, y) = y^2 - x^3 = 0$.



3. $f(x,y) = y^3 - x^3 - x^2 = 0$.



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We know what these look like due to some great computational power and the theory of cubic curves, but it's not easy to do this for general polynomials. We are going to discuss how to do this without the aid of a computer, and how to do it in real life completely accurately. To do so, we are going to discuss how to build a physical machine to do it for us.

2 Linkages

Definition 1. An abstract marked linkage (L, ℓ, W) is a finite graph L with a positive functions $\ell : E(L) \to \mathbb{R}_{>0}$ on the edges of L. The set $W \subseteq V(L)$ is a subset of the vertices of L and they are called the fixed vertices. A choice of W is called a marking. If $W = \varnothing$, then we say (L, ℓ) is an abstract linkage. Often times, we just write L for (L, ℓ, W).

The function ℓ assigns a positive length to each of the edges. Note that we do not require a graph L to be a metric graph. For example, consider $L = K_3$, the complete graph on 3 vertices. This looks like a triangle, but we can assign weights 1, 1, and 3 to the edges.

Definition 2. Let L be an abstract marked linkage as above. We say $\phi : V(L) \to \mathbb{R}^2$ is a planar realization of L if for any edge e connecting a vertex a and b,

$$\|\phi(a) - \phi(b)\| = \ell(e)^2.$$

If $W = \{w_1, \ldots, w_n\}$, let $Z = (z_1, \ldots, z_n) \in \mathbb{C}^n \cong \mathbb{R}^{2n}$. We define $C(L, Z)$ to be the set of all planar realizations of L such that $\phi(w_i) = z_i$ for $i = 1, \ldots, n$. The set $C(L, Z)$ is called the configuration space of L based at Z.

The set $C(L, Z)$ has a natural geometric structure since it can be viewed as a subset of $\mathbb{R}^n$. Given a vertex $v_i \in V(L)$, we are just looking at all ways $\phi(v_i) = (x_i, y_i)$. Then,

$$\|\phi(v_i) - \phi(v_j)\| = (x_i - x_j)^2 + (y_i - y_j)^2 = \ell([v_i v_j])^2.$$

These quadratic equations determine a space (a subvariety) in $\mathbb{R}^{2k}$ where k is the number of vertices.

Example 3. Consider the linkage L consisting of two vertices $V(L) = \{v_1, v_2\}$ and one edge $E(L) = \{[v_1 v_2]\}$. Let our weight be $\ell([v_1 v_2]) = 1$ and let our marking be $W = \{v_1\}$. Let $Z = \{(0,0)\}$. Then, $C(L, Z)$ is given by the unit circle. This is because $C(L, Z) \subseteq \mathbb{R}^4$ where $\phi(v_1) = (x_1, y_1) = (0,0)$ and $\phi([v_1 v_2])^2 = 1 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = x_2^2 + y_2^2$.

Example 4. Consider

where the first and last vertices are marked and sent to $(0,0)$ and $(2,0)$. If v is the vertex in the middle, our space $C(L, Z)$ is given by the set of all $(x, y) \in \mathbb{R}^2$ such that

$$x^2 + y^2 = (x-2)^2 + y^2 = 1.$$

Note that this implies that

$$\begin{aligned} x^2 &= (x-2)^2 = x^2 - 4x + 4 \\ 4x &= 4 \\ x &= 1. \end{aligned}$$

Then, plugging in $x = 1$ into our original equation tells us that $y^2 = 0$. This confirms what we already know - there is only one configuration (the one where v is sent to $(0,0)$). But, the fact that this y^2 appears tells us that our linkage has a bit of wiggle room. For those familiar with algebraic geometry, this is telling us that our $C(L, Z)$ as a scheme is non-reduced.

We often assume that our linkage is based at the origin, and is rotation-free. When doing so, we express our configuration space as $M(L)$.

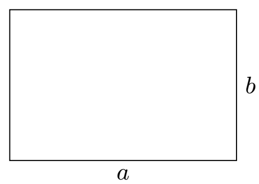
A remarkable theorem of Thurston says that studying linkages will tell you things about manifolds!

Theorem 5 - Thurston. Let X be a smooth compact manifold. Then, there is some linkage L such that $M(L)$ is diffeomorphic to a disjoint union of X.

We therefore have a natural question:

Question 6. Is $M(L)$ a always a smooth manifold?

The answer is no, in general. Consider the linkage



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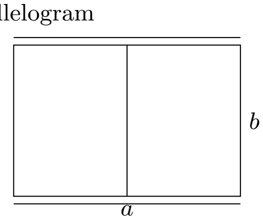
The space $M(L)$ is given by a union of two circles intersecting at two distinct points if $a \neq b$, and if $a = b$ it is three copies of $\mathbb{P}^1$ each intersecting one another at one distinct point. Both of these spaces are clearly not smooth, and we can see this by computing the tangent spaces at the points of intersection. We can fix this by rigidifying our parallelogram



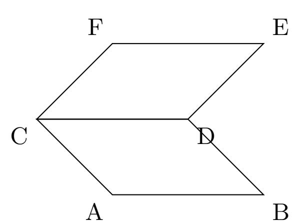
and this prevents our non-convex configurations from being possible.

I am claiming to you that I can create a machine that will draw for you a polynomial, so I had better explain how to encode polynomials in linkages.

Definition 7. Let $f : \mathbb{R}^{2m} \to \mathbb{R}^{2n}$ be a function. We define a functional linkage (L, ℓ, W, I, O) to be an abstract marked linkage with two additional sets of vertices I and O, where $I = \{P_1, \ldots, P_m\}$ and $O = \{Q_1, \ldots, Q_n\}$ satisfies the following condition: for any $\phi \in C(L, Z)$, we have that $f(\phi(P_1), \ldots, \phi(P_m)) = (\phi(Q_1), \ldots, \phi(Q_n))$.

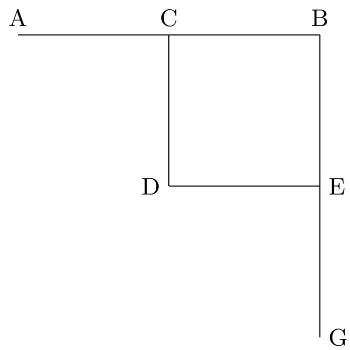
This notion is capturing the idea that we are sticking a pen at the ends of the vertices $Q_1, \ldots, Q_n$ and letting the pens trace out the path given by the linkage.

Example 8 - The translator. Let $f(x) = x + b$ where $b \in \mathbb{R}^2$ is a constant. Let L be the linkage given by



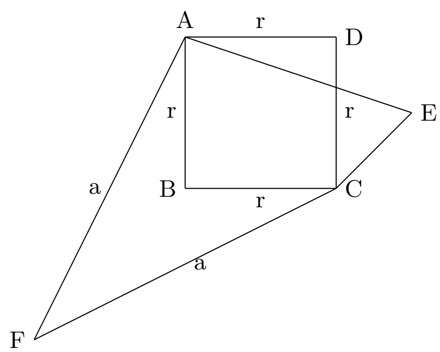
where $\ell([AC]) > \ell([CE])$. The input is F and the output is E. We let $W = \{A, B\}$ be the marking, and $Z = (0, b) \in (\mathbb{R}^2)^2$ be where the marking is sent to. Then, L is a functional linkage for $f(x) = x + b$ where $b \in \mathbb{C} \cong \mathbb{R}^2$. In order to get a functional linkage for $g(x) = x - b$, we just make E the input and F the output.

Example 9 - The scalar/The adder. Let $\lambda > 1$ and $f(z) = \lambda z$. Let L be the linkage given by



where A is mapped to $(0, 0)$ and D is our input and G is our output. Here, $\ell([AB]) = \lambda\ell([AC])$ and $\ell([BG]) = \lambda\ell([BE])$. Note that if we want a functional linkage for $z \mapsto z/\lambda$, we just need to make G the input and D the output. Here, the parallelogram $BCDE$ is rigifdified. To get the functional linkage for $f(z, w) = (z + w)/2$, we take L but do not fix any vertices and let $\{A, B\}$ be our inputs and D be our output.

Example 10 - The inversor. Let $f(z) = t^2/\bar{z}$ where t is a constant. Let $r, a > 0$ such that $a^2 - r^2 = t^2$. We consider the linkage L given by



where F is sent to 0. The vertex B is our input and D is our output.

We now discuss function composition in terms of linkages.

Proposition 11 - Function composition. Let L_f and L_g be functional linkages for $f : k^n \to k^m$ and $g : k^m \to k^r$. Then, there is a functional linkage L for the composition $g \circ f : k^n \to k^r$ given by taking the linkages L_f and L_g and hooking the output vertices of L_f to the input vertices of L_g. We glue the fixed vertices of L_f to the fixed vertices of L_g. Then, the resulting linkage L has input vertices given by the input vertices of L_f and the output vertices of L_g.

For experts, we are just taking the fibered product of $C(L_f, Z)$ with $C(L_g, Z)$.

We now have all the tools we need to talk about polynomials in general.

Theorem 12. Let $f(z, \bar{z}) : \mathbb{C} \to \mathbb{R}$ be a polynomial in two variables. Then, there is a linkage L which draws any bounded region of $f(x, y) = 0$.

Proof. We begin by constructing a functional linkage for $g(z, w) = zw$. We can construct a functional linkage for $z \mapsto z^2$ by noting that

$$\frac{1}{\bar{z} - 1/2} - \frac{1}{\bar{z} + 1/2} = \frac{1}{\bar{z}^2 - 1/4}.$$

Then, by composing with the inversion and translation linkages, we obtain a linkage for $z \mapsto z^2$. Now, we note that

$$zw = \frac{1}{2}\left((z+w)^2 - (z^2 + w^2)\right)$$

and the right-hand-side is given by a composition of linkages we know how to construct.

For a general polynomial $f(z, \bar{z}) : \mathbb{C} \to \mathbb{R}$, this is given by the iterative composition of addition, multiplication, and translation. So, we get a linkage L which is functional for f. Finally, we attach the output of L to 0. Then, the input vertex draws out the zero locus $f(z, \bar{z}) = 0$. $\square$