# Non-Abelian Hodge Theory

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**Abstract**

These are notes for a talk on Non-Abelian Hodge Theory for Elden's Seminar. I want to talk about the Riemann-Hilbert correspondence and the correspondence between Higgs bundles and local systems. In particular, I want to talk about $\mathbb{C}$-VHS and how it relates to the $\mathbb{C}^*$ on the moduli space of Higgs bundles. I then want to talk about parabolic Higgs bundles in the case of curves, especially when $X = \mathbb{P}^1 \setminus D$.

## 1 From first year topology to non-abelian cohomology

Let $X$ be a compact Kähler manifold (for example, any smooth projective variety over $\mathbb{C}$). We have the following isomorphisms:

$$\mathrm{Hom}(\pi_1(X), \mathbb{C})//\mathbb{C} \cong H^1(X, \mathbb{C}) \cong H^1_{\mathrm{dR}}(X) \cong H^1_{\mathrm{Dol}}(X) \cong H^{1,0}(X) \oplus H^{0,1}(X) \cong H^0(X, \Omega^1_X) \oplus H^1(X, \mathscr{O}_X).$$

The first isomorphism comes from the Universal Coefficient Theorem, the isomorphism between de Rham and Dolbeault cohomology comes from the Kähler-ness of the manifold, and the last isomorphism comes from the general result in Hodge theory that $H^q(X, \Omega^p_X) = H^{p,q}(X)$. A class in cohomology $[\alpha]$ is therefore equivalent to a 1-cocycle (gluing data for a rank 1 bundle) and a choice of 1-form on $X$.

We seek to study a non-abelian analogue of the phenomenon above, meaning the coefficients are some non-abelian group. The most natural group, is of course, some matrix group. For now, we set $G = \mathrm{GL}_n(\mathbb{C})$ but if we let $G$ be a subgroup (say, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{SL}_2$, etc. we get interesting behavior as well).

We define the following spaces:

**Definition 1.1.** Let $\mathrm{M}_{\mathrm{Betti}}(X, r) = \mathrm{Hom}(\pi_1(X), \mathrm{GL}_r(\mathbb{C}))// \mathrm{GL}_r(\mathbb{C})$. This is the character variety of $\pi_1(X)$. Let $\mathrm{M}_{\mathrm{dR}}(X, r)$ be the moduli space of flat vector bundles on $X$.

**Theorem 1.2 - (Riemann-Hilbert).** There are homeomorphisms $\mathrm{M}_{\mathrm{Betti}}(X, r) \cong \mathrm{M}_{\mathrm{dR}}(X, r) \cong \mathbb{L}$, where $\mathbb{L}_r$ is the space of local systems on $X$ of rank $r$.

*Proof.* The maps are as follows:

1. $f : \mathrm{M}_{\mathrm{dR}}(X, r) \to \mathbb{L}_r$ is given by $f : (V, \nabla) \mapsto \ker(\nabla)$ with inverse $\mathbb{V} \mapsto (\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_X, \mathrm{id} \otimes d)$.
2. $\mathrm{M}_{\mathrm{dR}}(X, r) \to \mathrm{M}_{\mathrm{Betti}}(X, r)$ is given by monodromy of solutions. **This is a highly transcendental operation, and I am not aware of any general method of writing out this map in coordinates. Any special cases worked out (besides $r = 1$) would be extremely interesting!** Best to sketch a picture here. Make sure to mention that monodromy data around a puncture is given by a conjugacy class because $\pi_1(X)$ is defined only up to conjugacy prior to picking a base point, and that complex analysis can only detect homotopy classes of loops.

□

**Remark 1.3.** $\mathrm{M}_{\mathrm{Betti}}$ and $\mathrm{M}_{\mathrm{dR}}$ are meant to be non-abelian cohomology theories, but they do not have a known group structure. These are only pointed spaces.

The story for abelian cohomology tells us that there should be a fourth space $\mathrm{M}_{\mathrm{Dol}}(X, r)$ which is analogous to Dolbeault cohomology and is homeomorphic to the first three. Points in this space should consists of pairs $(E, \theta)$ where $E$ is some sort vector bundle and $\theta$ is a 1-form.

## 2 Higgs bundles

The correct object to put inside $\mathrm{M}_{\mathrm{Dol}}(X, r)$ are rank $r$ Higgs bundles on $X$.

**Definition 2.1.** A Higgs bundle $(E, \theta)$ on $X$ is a (holomorphic) vector bundle $E$ with a $\mathscr{O}_X$-linear map $\theta : E \to E \otimes \Omega^1_X$ such that $\theta \wedge \theta = 0$.

We call a subbundle $F \subseteq E$ a sub-Higgs bundle if $\theta(F) \subseteq F \otimes \Omega^1_X$.

**Remark 2.2.** If $X$ is a smooth projective variety, GAGA tells us that $E$ is in fact an algebraic vector bundle and $\theta$ is an algebraic $\mathrm{End}(E)$-valued 1-form.

**Definition 2.3.** For a vector bundle $W$ on $X$, write $\mu(W) = c_1(W) \cdot [\omega]^{\dim X - 1} / \operatorname{rank} W$ to be the slope of $W$. A vector bundle $V$ is stable (resp. semistable) if for all proper nonzero subbundles $W \subsetneq V$ we have that $\mu(W) < \mu(V)$ (resp. $\mu(W) \leq \mu(V)$).

We say a Higgs bundle $(E, \theta)$ is a stable Higgs bundle if $\mu(F) < \mu(E)$ for all sub-Higgs bundles $F \subseteq E$.

We say a Higgs bundle is polystable if it is the direct sum of stable Higgs bundles.

**Definition 2.4.** Let $\mathrm{M}_{\mathrm{Dol}}(X, r)$ be the rank $r$ polystable Higgs bundles such that $c_1(E) \cdot [\omega]^{\dim X - 1} = c_2(E) \cdot [\omega]^{\dim X - 2} = 0$.

**Theorem 2.5 - (Non-Abelian Hodge Theorem).** There is a homeomorphism (a $\mathbb{R}$-analytic isomorphism (this is meaningless) $\mathrm{M}_{\mathrm{Dol}}(X, r) \cong \mathrm{M}_{\mathrm{Betti}}(X, r)$. The stable Higgs bundles correspond precisely to the irreducible representations in $\mathrm{M}_{\mathrm{Betti}}(X, r)$.

*Proof.* Analysis through harmonic bundles. The $c_1(E) \cdot [\omega]^{\dim X - 1} = c_2(E) \cdot [\omega]^{\dim X - 2} = 0$ condition is to ensure that certain metrics exist to make the analysis work. □

**Example 2.6.** Let $C$ be a smooth proper curve of genus $g$ and let $r = 1$. Then, $\mathrm{M}_{\mathrm{Dol}}(c, 1) \cong \mathrm{Jac}(C) \times H^0(X, \Omega^1_X) \cong (S^1)^{2g} \times \mathbb{C}^g$. On the other hand, $\mathrm{M}_{\mathrm{Betti}}(C, 1) = \mathrm{Hom}(H_1(X), \mathbb{C}^*) = \mathrm{Hom}(\mathbb{Z}^{2g}, \mathbb{C}^*) = (\mathbb{C}^*)^{2g}$. We know that

$$(S^1)^{2g} \times \mathbb{C}^g \cong (S^1)^{2g} \times (\mathbb{R})^{2g} \cong (S^1 \times \mathbb{R})^{2g} \cong (\mathbb{C}^*)^{2g}$$

and hence $\mathrm{M}_{\mathrm{Betti}}(C, 1) \cong \mathrm{M}_{\mathrm{Dol}}(C, 1)$ as expected.

## 3 Variations of Hodge structure and $\mathbb{C}^*$-actions

**Definition 3.1.** A $\mathbb{C}$-variation of Hodge structure on a complex manifold $S$ of weight $n$ is the data of a local system $\mathbb{V}$ on $S$ together with a decreasing filtration $F^\bullet V$ on $V = \mathbb{V} \otimes \mathscr{O}_S$ and a flat connection $\nabla : V \to V \otimes \Omega^1_X$ such that

1. on the fibers of $V$, the induced filtration $F^\bullet V_s$ on $V_s$ makes $V_s$ into a Hodge structure of weight $n$,
2. (Griffiths transversality) - for all $p$, we have that $\nabla(F^p V) \subseteq F^{p-1} \otimes \Omega^1_X$.

**Remark 3.2.** The Griffiths transversality condition is there because if $\mathbb{V} = R^i \pi_* \underline{\mathbb{C}}$ for some smooth proper map $\pi : X \to S$, Griffiths transversality is always satisfied.

**Definition 3.3.** Let $(V, F^\bullet, \nabla)$ be a $\mathbb{C}$-VHS on $S$. Then, write $E^p = F^p V / F^{p-1} V$ and $\theta_p : E^p \to E^{p-1} \otimes \Omega^1_X$ the induced map from $\nabla$. Then, set $E = \oplus E_p$ and $\theta = \oplus \theta_p$. We call $(E, \theta)$ the Higgs bundle induced from $(V, F^\bullet, \nabla)$. If $(E, \theta)$ is a Higgs bundle which comes from this contstruction, we say that $(E, \theta)$ is a system of Hodge bundles.

There is an action of $\mathbb{C}^*$ on $\mathrm{M}_{\mathrm{Dol}}(X, r)$ given by $t \cdot (E, \theta) = (E, t\theta)$. Note that in the graded case (i.e., $(E, \theta)$ is a system of Hodge bundles), $t$ acts on $E^p$ by $t^p$.

**Lemma 3.4.** Let $(E, \theta) \cong (E, t\theta)$ for some $t \in \mathbb{C}^*$ such that $t$ is not a root of unity. Then, $E$ has the structure of a system of Hodge bundles.

*Proof.* Let $f : E \to E$ be an automorphism such that $f\theta = t\theta f$. Then the characteristic polynomial of $f$ is given by $p(z) = z^r + a_1 z^{r-1} + \cdots + a_r$ where $a_j = (-1)^j \operatorname{tr}(\wedge^j f)$ where $r = \operatorname{rank} E$. But since $X$ is a compact complex manifold, the $\operatorname{tr}(\wedge^j f)$ are all constant. Hence, $p(z)$ has constant eigenvalues. We can then write $E = \oplus E_\lambda$ where $\lambda$ are the roots of $p(z)$ and $E_\lambda = \ker(f-\lambda)^n$ are the generalized Eigenspaces of $f$. Then,

$$(f - t\lambda)^n \theta = t^n \theta (f - \lambda)^n$$

so $\theta$ maps the eigenspace $E_\lambda$ to the $E_{t\lambda}$ eigenspace. So, we get eigenspaces for $\lambda, t\lambda, \ldots, t^s\lambda$, with $t^{-1}\lambda$ and $t^{s+1}\lambda$ not eigenvalues. (Here, we are using the fact that $t$ is not a root of unity.) Then setting $E^p = E_{t^{s-p}\lambda}$ we have that $\theta(E^p) \subseteq E^{p-1} \otimes \Omega^1_X$. Therefore we get the structure of a system of Hodge bundles. □

Therefore, systems of Hodge bundles ($\mathbb{C}$-VHS), are precisely the fixed points of the $\mathbb{C}^*$-action. We use this description to study certain special points in the various moduli spaces.

**Corollary 3.5.** Let $X$ and $Y$ are compact Kähler manifolds and $f : Y \to X$ is a map such that $f_* : \pi_1(Y) \to \pi_1(X)$ is surjective. If $V$ is a bundle such that $f^*V$ comes from a $\mathbb{C}$-VHS on $Y$, then $V$ comes from a $\mathbb{C}$-VHS on $X$. An example where this theorem applies is when, for example, $Y$ is a hyperplane section of $X$. Then the Lefschetz hyperplane theorem tells us that we have an injection $H^{n-1}(X, \mathbb{Z}) \to H^{n-1}(Y, \mathbb{Z})$ which corresponds to a surjection on fundamental groups.

*Proof.* The action of $\mathbb{C}^*$ commutes with $f^*$. If there are two local systems (representations) $V_1$ and $V_2$ such that $f^*V_1 \cong f^*V_2$, then $V_1 \cong V_2$ since $f$ is surjective on fundamental groups. Let $V_t$ be the local system given by the action of $t$ on $V$. Then since $f^*V$ is a $\mathbb{C}^*$-fixed point, we know that $f^*V \cong f^*V_t$ and so $V \cong V_t$. Therefore, $V$ is a $\mathbb{C}^*$-fixed point and comes from a $\mathbb{C}$-VHS. □

**Definition 3.6.** Let $G$ be a reductive algebraic group. A representation $\rho : \pi_1(X) \to G$ is called rigid if it is an isolated point of $\mathrm{Hom}(\pi_1(X), G)//G$. Equivalently, the local monodromy data of $X$ uniquely determines the representation up to isomorphism.

**Corollary 3.7.** Any rigid representation comes from a complex variation of Hodge structure.

*Proof.* Let $(E, \theta)$ be a Higgs bundle coming from a rigid representation. Let $t_i$ be a sequence of elements of $\mathbb{C}^*$ such that none of the $t_i$ are roots of unity, and $\lim t_i = 1$. Then, $\lim_{i\to\infty}(E, t_i\theta) = (E, \theta)$.

Since $(E, \theta)$ is rigid as a representation of $G$, so $(E, \theta) \cong (E, t_n\theta)$ for some $t_n$ as $t_i \to 1$. Therefore, by Lemma $t$, we know that $(E, \theta)$ is a system of Hodge bundles and hence comes from a $\mathbb{C}$-VHS. □

**Lemma 3.8.** Suppose $X$ is a smooth projective variety, and $G$ a reductive complex algebraic group. Any representation $\rho : \pi_1(X) \to G$ can be deformed to a representation which comes from a $\mathbb{C}$-VHS.

*Proof.* We invoke the following fact: the map $h : \mathrm{M}_{\mathrm{Dol}}(X, r) \to C$ (where $C$ is the space of polynomials with coefficients in $\mathrm{Sym}^\bullet \Omega^1_X$) given by $h(E, \theta) = p_\theta(z)$ where $p_\theta(z)$ is the characteristic polynomial of $\theta$, is proper.

Then, we take the limit $\lim_{t\to 0}(E, t\theta) = (E', \theta')$. Such a limit exists, since $\lim_{t\to 0} h(t\theta)$ approach $z^r$ and then by properness of $h$, we get a limit $(E', \theta')$. This limit is unique and hence is preserved by $\mathbb{C}^*$. □

**Remark 3.9.** In the case of smooth projective curves of genus $g$, whenever $g > 0$ we never have rigid representations. This is because given a presentation of $\pi_1(C) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g | \prod [a_i, b_i] \rangle$ and some conjugacy classes for each generator $C_1, \ldots, C_{2g}$, any representation (given by matrices $A_1, \ldots, A_{2g}$ respecting the relations) can be deformed by considering $\lambda A_1, \ldots, \lambda A_{2g}$ for $\lambda \in \mathbb{C}^*$. This is not conjugation.

# 4 The parabolic case for curves

In the case where $X$ is not a projective variety, we no longer know that any holomorphic vector bundle is an algebraic vector bundle since we cannot use GAGA. However, Deligne provides for us a solution - a holomorphic vector bundle with flat connection does not have a unique algebraic structure, but it has a *canonical* algebraic bundle structure with a regular flat connection.

**Definition 4.1.** Let $X$ be smooth and $D$ a simple normal crossings divisor (locally looks like $V(x_1 \cdots x_n)$ and each of its components are smooth). We define a logarithmic form $p$-form $\omega$ with respect to $D$ to be an algebraic $p$-form on $X \setminus D$ such that $\omega$ and $d\omega$ have poles of order at most 1 along $D$.

We set $\Omega^p_X(\log D)$ to be the sheaf of logarithmic $p$-forms with respect to $D$.

**Example 4.2.** We have a short exact sequence

$$0 \to \Omega^1_X \hookrightarrow \Omega^1_X(\log D) \to \oplus (i_j)_* \mathscr{O}_{D_j} \to 0$$

where $D = \sum D_j$ and $i_j$ are the inclusions of $D_j$.

On a curve $C$, if $D = x_1 + \cdots + x_n$, our logarithmic 1-forms are locally of the form $f(z)\frac{dz}{z - x_i}$ as expected.

From now on, we assume $X = C$ is a curve for simplicity.

**Definition 4.3.** A parabolic bundle $E_*$ with respect to a divisor $D = x_1 + \cdots + x_n$ is a vector bundle $E$ with the data

1. a flag $0 = E^0_{x_i} \subseteq \cdots \subseteq E^{n_j}_{x_i} = E_{x_i}$
2. a sequence of real numbers $0 \leq \alpha^1_i < \cdots < \alpha^{n_j}_i < 1$.

**Definition 4.4.** Given a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$, we set the residue at $x_i$ to be the matrix $\mathrm{Res}(\nabla)(x_i) \in \mathrm{End}(E)_{x_i}$. Let $\eta^1_i, \ldots, \eta^{n_i}_i$ be the eigenvalues of $\mathrm{Res}(\nabla)(x_i)$. Then we define

$$\lambda^j_i = \mathrm{Re}(\eta^j_i) - \lfloor \mathrm{Re}(\eta^j_i) \rfloor \in [0, 1).$$

We reorder the $\lambda^j_i$ so that $0 \leq \lambda^1_i < \cdots < \lambda^{n_i}_i < 1$. We let $E^j_{x_i}$ be the direct sums of the generalized eigenspaces of the eigenvalues $\eta^j_i$ and we set $E_*$ to be the parabolic bundle with these flags and the weights $\lambda^j_i$.

**Remark 4.5.** The idea behind this $\lambda^j_i \in [0, 1)$ condition is that the flat bundle with connection $\nabla = d + \lambda\frac{dz}{z}$ on $\mathbb{A}^1 \setminus \{0\}$ which has residue $\lambda$. Then, the monodromy representation is given by $1 \mapsto e^{2\pi i \lambda}$. Note that there are many flat connections with this local monodromy - $d + (\lambda + 1)\frac{dz}{z}$, for example. However, requiring that $\lambda \in [0, 1)$ ensures that we get a unique flat bundle with logarithmic singularities.

**Definition 4.6.** We set the parabolic degree of a parabolic bundle $E_*$ to be

$$\text{par-deg}(E_*) = \deg E + \sum_{i=1}^{n} \sum_{k=1}^{n_i} \alpha^k_i \cdot \dim(E^k_{x_i}/E^{k-1}_{x_i}).$$

We set the parabolic slope to be $\mu_*(E_*) = \text{par-deg}(E_*)/\operatorname{rank} E$, and we call $E_*$ parabolic stable if for all subbundles $F_*$, $\mu_*(F_*) < \mu_*(E_*)$. If $E_*$ is a parabolic Higgs bundle (a pair $(E_*, \theta)$ where $\theta : E \to E \otimes \Omega^1_X(\log D)$ is a $\mathscr{O}_X$-linear map), we call it stable if stability holds with respect to all sub-Higgs bundles.

**Theorem 4.7.** Let $C$ be a smooth proper curve and $D$ a reduced effective divisor on $C$. There is a homeomorphism of moduli spaces $\mathrm{M}_{\mathrm{Betti}}(C \setminus D, r) \cong \mathrm{M}_{\mathrm{dR}}(C \setminus D, r) \cong \mathrm{M}_{\mathrm{Dol}}(C \setminus D, r)$ where $\mathrm{M}_{\mathrm{dR}}(C \setminus D, r)$ is the moduli space of flat logarithmic bundles on $C$ with residues lying in $[0, 1)$, and $\mathrm{M}_{\mathrm{Dol}}(C \setminus D, r)$ is the moduli space of polystable parabolic Higgs bundles.

**Theorem 4.8.** All of the results in section 3 (on $\mathbb{C}$-VHS) hold in the parabolic case.