Six Functor Formalisms

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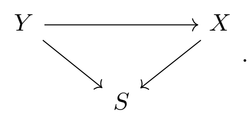
Abstract

We discuss six functor formalisms, especially within the context of Étale cohomology

1 Introduction

Grothendieck taught us that we should always think about geometry in a relative context: that is, we should be studying categories of S-schemes for some base S. That is, our objects are of the form $X \to S$ and our morphisms are commutative triangles



Consequently, if we wish to study $H^\bullet(X)$, where $H^\bullet$ is some (reasonable) cohomology theory, we then may wish to study cohomology *in a relative way*. That is, to study $H^\bullet(X)$ we may want to study all morphisms $f^* : H^\bullet(Y) \to H^\bullet(X)$ (and the same for f_*, derived pushforwards, $f_!$, etc).

For this reason, even if X is very nice (say smooth, projective, irreducible), we still might want to study the cohomology of non-proper spaces.

Example 1. If $Y = *$ is a point, then $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(*)$ coincides with the global sections functor, since $f_*F(*) = F(f^{-1}(*)) = F(X) = \Gamma(X, F)$. Then, taking higher derived functors yields $R^n f_* F = H^n(X, F)$ and therefore we obtain sheaf cohomology through the relative perspective.

Example 2. If $Y = *$ is a point, then $f_! F$ recovers compactly supported global sections of F. Taking higher derived functors then gives us compactly supported cohomology: $R^n f_! F = H_c(X, F)$.

Since we are studying derived functors, we will need to pass to derived categories.

2 Categorical considerations

Definition 3. Let $\mathcal{A}$ be an abelian category. We write $K(\mathcal{A})$ to be the category of chain complexes $\mathcal{A}$. That is,

1. the objects of $K(\mathcal{A})$ are chain complexes $(K^\bullet, d^\bullet)$ where for all i, K^i is an object of $\mathcal{A}$.
2. the morphisms are chain morphisms $f^\bullet : (C^\bullet, d^\bullet) \to (K^\bullet, d^\bullet)$.

We let $D(\mathcal{A})$ be the localization of $K(\mathcal{A})$ with respect to quasi-isomorphisms. The subcategory $D^b(\mathcal{A})$ of $D(\mathcal{A})$ consists of those chain complexes with only finitely many nonzero terms. We write $D^+(\mathcal{A})$ to be the subcategory consisting of objects $(K^\bullet, d^\bullet)$ such that $K^n = 0$ for some $n < 0$.

Definition 4. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of abelian categories. Let $i : D^+(\mathcal{A}) \to K^+(\mathrm{Inj}(\mathcal{A}))$ be the functor sending a chain complex to an injective resolution of the chain complex. This is an equivalence of categories, and hence we can consider the composition

$$D^+(\mathcal{A}) \to K^+(\mathrm{Inj}(\mathcal{A})) \to K^+(\mathcal{B}) \to D^+(\mathcal{B}).$$

This composition is the total derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$.

Example 5. The total derived functor RF is related to $R^n F$ via the fact $R^n F(X) = H^n(RF(X))$.

We are primarily interested in the case when $\mathcal{A}$ is some reasonable category of sheaves.

Example 6. The functor $Rf_!$ does not always admit an adjoint. But if X and Y are locally compact Hausdorff spaces, then we get an adjoint $f^! : D(\mathrm{Sh}(Y)) \to D(\mathrm{Sh}(X))$.

Definition 7. Given a morphism $f : X \to Y$ (in good situations), we have the following six functors: $Rf_*, Lf*, Rf_!, f^!, - \otimes -$ and $\mathrm{Hom}(-,-)$.

3 The six operations

Definition 8. Let $(C(X), \otimes, \mathrm{Hom})$ be a triangulated tensor category (I suggest you look this up on nlab. The definition is long. For our purposes, you can replace $C(X)$ with $D(\mathrm{Sh}(X))$ or the derived category of constructible ℓ-adic sheaves, etc.) Given a morphism $f : X \to Y$, we get adjunctions $f^* \dashv f_*$ and $f_! \dashv f^!$ and we have a natural transformation $f_! \to f_*$. We call the functors $\otimes$, Hom, f_*, f^*, $f_!$, and $f^!$ Grothendieck's six operations.

Note that we want f^* to respect tensor products, much like in the case of vector bundles.

Example 9. Let $p : X \to B$ be a morphism. Then for any $F \in C(X)$, $H^\bullet(X, F) = p_*F \in C(B)$ and $H_c^\bullet(X, F) = p_!F \in C(B)$. To recover the honest cohomology groups, we have that

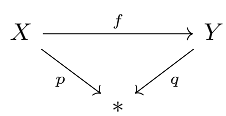
$$H^n(X, F) = \hom_{C(B)}(\mathbb{I}, p_*F[n])$$
$$H_c^n(X, F) = \hom_{C(B)}(\mathbb{I}, p_!F[n])$$

where $\mathbb{I}$ is the unit object in $C(B)$ and $[n]$ denotes shifting the indexing by n.

When $F = \mathbb{I}$, we just write then $H^\bullet(X)$ and $H_c^\bullet(X)$.

Example 10. Let k be a field such that ℓ is invertible, and $\mathrm{cd}_\ell(k) < \infty$. Then, $C(X) = D_c^b(X, \mathbb{Q}_\ell)$ (the bounded derived category of constructible ℓ-adic sheaves) gives us ℓ-adic cohomology.

Example 11. Consider the relative situation



Then since f^* and f_* form an adjoint pair we get that there is a morphism $\mathrm{id} \to f_*f^*$ and hence a morphism $\eta : q_* \to q_*f_*f^* = p_*f^*$. We then get a morphism on cohomology

$$H^\bullet(Y, F) = H^\bullet(X, f^*F).$$

Similarly, by adjunction we have a map $q_! \to q_!f_*f^*$. If f is proper, we have an isomorphism $f_* \cong f_!$ and so $q_!f_*f^* = q_!f_!f^* = p_!f^*$. Then, we get a morphism

$$H_c^\bullet(Y, F) \to H_c^n(X, f^*F).$$

Example 12. Let $p : X \to B$ be a morphism. Then since p^* respects tensor products, p_* sends commutative algebras in $C(X)$ to commutative algebras in $C(B)$. (A commutative algebra in $C(X)$ is an object K such that $K \otimes K \to K$ gives K the structure of a commutative algebra.)Then, $H^\bullet(X) = p_*\mathbb{I} \in C(B)$ has the structure of a commutative algebra, from which we get the cup product on cohomology.

4 Duality

Example 13. Let X be a smooth manifold of dimension d, and let $f : X \to *$ be the unique map. Let R be a ring. Let $C(X)$ be the category of singular cochains (with quasi-isomorphisms localized). Then, $f^!\mathbb{I} = \omega_{X,\mathbb{Z}}[d]$ is the shifted $\mathbb{Z}$-orientation sheaf. Then, X is orientable if and only if $\omega_{X,R}$ is the constant sheaf R.

Example 14. Let X be a smooth k-variety of dimension d. Let $C(X)$ be the derived category of (bounded) coherent sheaves on X. If $f : X \to \mathrm{Spec}(k)$, then $f^! k \cong \omega_X[d]$ is the shifted canonical sheaf on X.

Example 15. Let X be a smooth k-variety of dimension d, and ℓ a prime invertible in k. Let $f : X \to \mathrm{Spec}(k)$ be the structure morphism. Let $C(X)$ be the derived category of ℓ-adic sheaves on X. Then $f^! \mathbb{Q}_\ell = \mathbb{Q}_\ell(d)[2d]$ where $\mathbb{Q}_\ell(d)$ is the d-th Tate twist.

Theorem 16. Let the assumptions of example 13 to 15 hold in the following situations, respectively:

1. (Poincaré duality): if X is orientable, then $H^n_c(X, \mathbb{Q})^\vee \cong H^{d-n}(X, \mathbb{Q})$.
2. (Serre duality): if X is proper, $H^n(X, \mathscr{O}_X)^\vee \cong H^{d-n}(X, \omega_X)$.
3. (Poincaré duality in the ℓ-adic setting): $H^n_c(X, \mathbb{Q}_\ell)^\vee \cong H^{2d-n}(X, \mathbb{Q}_\ell(d))$.

Proof. Since either X is proper or we are dealing with compactly supported cohomology in all three situations, the following computation suffices to prove all three simultaneously.

We have that after dualizing,

$$H^n_c(X)^\vee = \mathrm{hom}(f_! f^* \mathbb{I}[n], \mathbb{I}).$$

Then by adjunction,

$$\mathrm{hom}(f_! f^* \mathbb{I}[n], \mathbb{I}) = \mathrm{hom}(\mathbb{I}, f_* f^! \mathbb{I}[-n])$$

which is then each of the desired cohomology groups, when replacing $f^! \mathbb{I}$ with $\omega_{X,R}[d]$, $\omega_X[d]$, and $\mathbb{Q}_\ell(d)[2d]$. □

Observe that in each case, $f^! \mathbb{I}$ serves as our dualizing object. This is true in general - if we have a dualizing object, it will be $f^! \mathbb{I}$.